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Loops on surfaces, Feynman diagrams, and trees

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Abstract

We relate the author's Lie cobracket in the module additively generated by loops on a surface with the Connes–Kreimer Lie bracket in the module additively generated by trees.

Keywords: rooted trees; pre-Lie algebras; Connes-Kreimer coproduct

1. Introduction

In 1989, the author introduced for any oriented surface Σ, a Lie cobracket ν in the module $Z = Z(\Sigma)$ generated by the homotopy classes of loops on Σ, see [7,8]. The cobracket ν complements the Goldman Lie bracket in Z and makes Z into a Lie bialgebra in the sense of Drinfeld. One of the main results of [8] is an algebraic quantization of this Lie bialgebra in terms of a Hopf algebra of knots in $\Sigma \times \mathbb{R}$. The Goldman Lie bracket has a transparent geometric nature: it is a reformulation of the Poisson bracket determined by the symplectic structure on the Teichmüller space (and/or other similar moduli spaces). On the other hand, the geometric nature of the cobracket ν remained mysterious. We argue here that the reason for the existence of this cobracket is that generic loops on Σ can be viewed as Feynman diagrams (of a rather special type). More precisely, we relate ν to the pre-Lie algebras and Hopf algebras of rooted trees introduced by Connes and Kreimer [2] in their fundamental work on the algebraic foundations of the perturbative quantum field theory. We introduce

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similar algebras generated by generic loops on Σ and define their canonical projection to the Connes–Kreimer algebras. Sections 2–7 deal with the Lie and pre-Lie (co)algebras arising in this theory. Section 8 is devoted to the associated Hopf algebras. In the last Section 9, we consider similar algebraic structures in the realms of Wilson loops and knot diagrams on Σ. Throughout the paper, we fix a commutative ring with unit R. The symbol $\otimes$ denotes the tensor product of R-modules over R. The symbol Σ denotes a smooth oriented surface (possibly with boundary).

2. Lie and pre-Lie coalgebras

We recall the algebraic language of pre-Lie and Lie coalgebras used systematically in the paper.

2.1. Lie coalgebras

For an R-module L, denote by Perm_L the permutation $x \otimes y \mapsto y \otimes x$ in $L^{\otimes 2} = L \otimes L$ and by τ_L the permutation $x \otimes y \otimes z \mapsto z \otimes x \otimes y$ in $L^{\otimes 3} = L \otimes L \otimes L$. A Lie algebra over R can be defined as an R-module L endowed with an R-homomorphism (the Lie bracket) $\theta : L^{\otimes 2} \to L$ such that $\theta \circ \mathrm{Perm}_L = -\theta$ and (the Jacobi identity)

$$\theta \circ (\mathrm{id}_L \otimes \theta) \circ (\mathrm{id}_{L^{\otimes 3}} + \tau_L + \tau_L^2) = 0 \in \mathrm{Hom}_R(L^{\otimes 3}, L)$$

Here, for a set S we denote by id_S the identity mapping $S \to S$.

Dually, a *Lie coalgebra* over R is an R-module A endowed with an R-homomorphism (the Lie cobracket) $\nu : A \to A^{\otimes 2}$ such that $\mathrm{Perm}_A \circ \nu = -\nu$ and

$$(\mathrm{id}_{A^{\otimes 3}} + \tau_A + \tau_A^2) \circ (\mathrm{id}_A \otimes \nu) \circ \nu = 0 \in \mathrm{Hom}_R(A, A^{\otimes 3}) \tag{2.1.1}$$

A Lie coalgebra (A, ν) gives rise to the *dual Lie algebra* $A^* = \mathrm{Hom}_R(A, R)$, where the Lie bracket is the homomorphism $A^* \otimes A^* \to A^*$ adjoint to ν. For $a, b \in A^*$, the bracket $[a, b] \in A^*$ evaluates on $x \in A$ by

$$[a, b](x) = \sum_i a(x_i^{(1)}) b(x_i^{(2)}) \in R \tag{2.1.2}$$

for any (finite) expansion $\nu(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in A \otimes A$.

A *Lie coalgebra homomorphism* $(A, \nu) \to (A', \nu')$ is an R-linear homomorphism $f : A \to A'$ such that $\nu' f = (f \otimes f)\nu$. The adjoint map $f^* : (A')^* \to A^*$ is then a Lie algebra homomorphism.

2.2. Pre-Lie algebras and coalgebras

Pre-Lie algebras were introduced by Gerstenhaber [4] and Vinberg [11] independently. A (left) pre-Lie algebra over R is an R-module L endowed with an R-bilinear multiplication

$L \times L \to L$, denoted $\star$, such that for any $x, y, z \in L$,

$$(x \star y) \star z - x \star (y \star z) = (y \star x) \star z - y \star (x \star z) \tag{2.2.1}$$

There is a similar notion of right pre-Lie algebras. We will consider only left pre-Lie algebras and refer to them simply as pre-Lie algebras. Equality (2.2.1) implies that $[x, y] = x \star y - y \star x$ is a Lie bracket in L.

A *pre-Lie algebra homomorphism* $(L, \star) \to (L', \star')$ is an R-linear homomorphism $f : L \to L'$ such that $f(x \star y) = f(x) \star' f(y)$ for all $x, y \in L$.

Dualizing formula (2.1.1), we obtain a notion of a pre-Lie coalgebra. A (left) *pre-Lie coalgebra* is an R-module A endowed with an R-linear homomorphism $\rho : A \to A \otimes A$ such that

$$(\mathrm{id}_{A^{\otimes 3}} - P_A^{1,2})((\rho \otimes \mathrm{id}_A)\rho - (\mathrm{id}_A \otimes \rho)\rho) = 0 \in \mathrm{Hom}_R(A, A^{\otimes 3}) \tag{2.2.2}$$

where $P_A^{1,2}$ is the endomorphism of $A^{\otimes 3}$ permuting the first and second tensor factors. Given a pre-Lie coalgebra (A, ρ), the dual module $A^* = \mathrm{Hom}_R(A, R)$ acquires a structure of a pre-Lie algebra: for $a, b \in A^*$, the value of $a \star b \in A^*$ on $x \in A$ is given by the right-hand side of formula (2.1.2) for any expansion $\rho(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in A \otimes A$.

Lemma 2.2.1. *For any pre-Lie coalgebra* (A, ρ)*, the homomorphism* $\nu = \rho - \mathrm{Perm}_A \rho : A \to A^{\otimes 2}$ *is a Lie cobracket.*

Proof. It is obvious that $\mathrm{Perm}_A \circ \nu = -\nu$. Formula (2.1.1) follows from the identity

$$\begin{aligned}(\mathrm{id}_{A^{\otimes 3}} + \tau_A + \tau_A^2) \circ (\mathrm{id}_A \otimes \nu) \circ \nu = -(\mathrm{id}_{A^{\otimes 3}} + \tau_A + \tau_A^2) \circ (\mathrm{id}_{A^{\otimes 3}} - P_A^{1,2}) \\ \circ ((\rho \otimes \mathrm{id}_A)\rho - (\mathrm{id}_A \otimes \rho)\rho)\end{aligned}$$

which holds for *any* R-linear homomorphism $\rho : A \to A^{\otimes 2}$ and $\nu = \rho - \mathrm{Perm}_A \rho$. □

A *pre-Lie coalgebra homomorphism* $(A, \rho) \to (A', \rho')$ is an R-linear homomorphism $f : A \to A'$ such that $\rho' f = (f \otimes f)\rho$. The adjoint map $f^* : (A')^* \to A^*$ is then a pre-Lie algebra homomorphism. The reader should always keep in mind that a pre-Lie coalgebra homomorphism f is always a Lie coalgebra homomorphism of the associated Lie coalgebras and its adjoint f^* is a Lie algebra homomorphism of the dual Lie algebras.

3. Pre-Lie coalgebra of loops

We define a pre-Lie coalgebra of loops on a smooth oriented surface Σ.

3.1. Loops and Feynman diagrams

A *generic loop* on Σ is a smooth immersion $\alpha : S^1 \to \Sigma - \partial\Sigma$ having only double transversal self-crossings. The set of the self-crossings of α is denoted $\#\alpha$; it is always finite. We will consider only generic loops and refer to them simply as loops. The circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is oriented counterclockwise, this makes all loops oriented. A *pointed loop* is a pair (a loop α on Σ, a point $*_\alpha \in \alpha(S^1) - \#\alpha$). The latter point is called the *base point* of α.

Two pointed loops α, β on Σ can be obtained from each other by *reparametrization* if $*_\alpha = *_\beta$ and there is an orientation preserving homeomorphism $f : S^1 \to S^1$ such that $\beta = \alpha f$. Pointed loops α, β on Σ are *ambient isotopic* if there is a continuous family of homeomorphisms $\{h_t : \Sigma \to \Sigma\}_{t\in[0,1]}$ such that $h_0 = \mathrm{id}_\Sigma$, $\beta = h_1\alpha$, and $*_\beta = h_1(*_\alpha)$. We say that two pointed loops on Σ are *isotopic* if they can be obtained from each other by ambient isotopy and/or reparametrization. For example, slightly pushing $*_\alpha$ along $\alpha(S^1) - \#\alpha$ we obtain a pointed loop isotopic to $(\alpha, *_\alpha)$. It is clear that isotopy of loops is an equivalence relation. We shall usually identify loops with their isotopy classes.

We shall also consider loops which are only piecewise smooth. This should create no problem since the non-smooth points (looking like corners of a broken line) will be finite in number and distinct from crossing points. All such loops can be smoothed in the obvious way.

Any loop α on Σ gives rise to a Feynman diagram. It is formed by the circle S^1 and a set of straight segments $\{e_p\}_{p\in\#\alpha}$. The endpoints of e_p are the two (distinct) points of the set $\alpha^{-1}(p) \subset S^1$. The segments $\{e_p\}_p$ lie in the unit disk $D^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$ and have distinct endpoints but may meet inside D^2. We say that $p, q \in \#\alpha$ are *linked* if $e_p \cap e_q \neq \emptyset$ and *unlinked* if $e_p \cap e_q = \emptyset$. We shall see below that the segments $\{e_p\}_p$ can be naturally oriented. The Feynman diagram $(S^1, \{e_p\}_p)$ is well known in knot theory as the *Gauss diagram* of α. In the physical language, the circle S^1 represents a fermion and the segments $\{e_p\}_p$ represent photons. Note that in this picture the photons do not interact with each other.

3.2. Pre-Lie comultiplication for loops

Let $\mathcal{L} = \mathcal{L}(\Sigma)$ be the R-module freely generated by the set of isotopy classes of pointed loops on Σ. Elements of $\mathcal{L}$ are finite formal linear combinations of such isotopy classes with coefficients in R. We define a pre-Lie comultiplication $\rho : \mathcal{L} \to \mathcal{L}^{\otimes 2}$.

We shall use the following notation: for two distinct points $P, Q \in S^1$, denote by PQ the oriented embedded arc in S^1 which starts at P, goes in the positive (counterclockwise) direction and terminates at Q. Clearly, $PQ \cup QP = S^1$ and $PQ \cap QP = \{P, Q\}$. For a loop $\alpha : S^1 \to \Sigma$, denote by $\alpha_{P,Q}$ the path in Σ obtained by restricting α to the arc PQ.

It suffices to define $\rho : \mathcal{L} \to \mathcal{L}^{\otimes 2}$ on the basis of $\mathcal{L}$. Consider the generator of $\mathcal{L}$ presented by a pointed loop $(\alpha : S^1 \to \Sigma, *_\alpha \in \alpha(S^1))$. This loop traverses every point $p \in \#\alpha$ twice in two different tangent directions. The set $\alpha^{-1}(p) \subset S^1$ consists of two points $p_1, p_2 \in S^1$ numerated so that starting at $\alpha^{-1}(*_\alpha) \in S^1$ and moving along S^1 counterclockwise we first meet p_1 and then p_2. The path α_{p_1,p_2} (resp. α_{p_2,p_1}) is a closed loop on Σ which starts off at p in one of the two tangent directions mentioned above and follows along α until the first return to p. The loop α_{p_2,p_1} goes through $*_\alpha$ and we take $*_\alpha$ as its base point. As the base point of α_{p_1,p_2} we take p. Set $\varepsilon_p = +1$ if the pair (the positive tangent direction of α at p_1, the positive tangent direction of α at p_2) is positive with respect to the orientation of Σ. In the opposite case, set $\varepsilon_p = -1$. Note that ε_p and the numeration p_1, p_2 depend on the choice of $*_\alpha$. Finally, set

$$\rho(\alpha, *_\alpha) = \sum_{p\in\#\alpha} \varepsilon_p(\alpha_{p_1,p_2}, p) \otimes (\alpha_{p_2,p_1}, *_\alpha) \tag{3.2.1}$$

Lemma 3.2.1. *The homomorphism $\rho : \mathcal{L} \to \mathcal{L}^{\otimes 2}$ is a pre-Lie comultiplication.*

Proof. We verify formula (2.2.2) for $A = \mathcal{L}$. Pick a pointed loop α on Σ.

A direct application of the definitions shows that both $(\rho \otimes \mathrm{id})\rho(\alpha)$ and $(\mathrm{id} \otimes \rho)\rho(\alpha)$ are sums of certain expressions numerated by pairs of unlinked crossings $p, q \in \#\alpha$. Pick such a pair p, q. Starting at $\alpha^{-1}(*_\alpha)$ and moving along S^1 counterclockwise, we meet the points p_1, p_2, q_1, q_2 in a certain order such that p_1 appears before p_2 and q_1 appears before q_2. Exchanging if necessary the letters p and q, we can assume that the first point we meet is p_1. Then, the order in question is either (i) p_1, p_2, q_1, q_2 or (ii) p_1, q_1, q_2, p_2. The contribution of p, q to $\rho(\alpha)$ is $\varepsilon_p(\alpha_{p_1,p_2}, p) \otimes (\alpha_{p_2,p_1}, *_\alpha) + \varepsilon_q(\alpha_{q_1,q_2}, q) \otimes (\alpha_{q_2,q_1}, *_\alpha)$. In the case (i), the contribution of p, q to $(\rho \otimes \mathrm{id})\rho(\alpha)$ is 0 and the contribution of p, q to $(\mathrm{id} \otimes \rho)\rho(\alpha)$ is

$$\varepsilon_p\varepsilon_q(\alpha_{p_1,p_2}, p) \otimes (\alpha_{q_1,q_2}, q) \otimes (\alpha_{p_2,q_1}\alpha_{q_2,p_1}, *_\alpha) + \varepsilon_p\varepsilon_q(\alpha_{q_1,q_2}, q) \otimes (\alpha_{p_1,p_2}, p) \otimes (\alpha_{q_2,p_1}\alpha_{p_2,q_1}, *_\alpha)$$

Here, $\alpha_{p_2,q_1}\alpha_{q_2,p_1}$ and $\alpha_{q_2,p_1}\alpha_{p_2,q_1}$ are the loops obtained as products of the paths α_{p_2,q_1} and α_{q_2,p_1}. Note that up to reparametrization $\alpha_{p_2,q_1}\alpha_{q_2,p_1} = \alpha_{q_2,p_1}\alpha_{p_2,q_1}$.

In the case (ii), the contributions of p, q to $(\rho \otimes \mathrm{id})\rho(\alpha)$ and $(\mathrm{id} \otimes \rho)\rho(\alpha)$ are both equal to

$$\varepsilon_p\varepsilon_q(\alpha_{q_1,q_2}, q) \otimes (\alpha_{p_1,q_1}\alpha_{q_2,p_2}, p) \otimes (\alpha_{p_2,p_1}, *_\alpha)$$

In both cases, the contribution of the pair p, q to $((\rho \otimes \mathrm{id})\rho - (\mathrm{id} \otimes \rho)\rho)(\alpha)$ is invariant under the permutation of the first and second tensor factors $P^{1,2}$ and is therefore annihilated by $\mathrm{id} - P^{1,2}$. This proves (2.2.2). □

The pre-Lie comultiplication ρ induces by antisymmetrization a Lie cobracket ν in $\mathcal{L}$. On the basis of $\mathcal{L}$,

$$\nu(\alpha, *_\alpha) = \sum_{p \in \#\alpha} \varepsilon_p((\alpha_{p_1,p_2}, p) \otimes (\alpha_{p_2,p_1}, *_\alpha) - (\alpha_{p_2,p_1}, *_\alpha) \otimes (\alpha_{p_1,p_2}, p)) \quad (3.2.2)$$

The resulting Lie coalgebra and the dual Lie algebra were introduced in [7,8], cf. Section 7.

4. Connes–Kreimer pre-Lie coalgebras

We recall (in a convenient form) and generalize the definitions of Connes and Kreimer.

4.1. Pre-Lie coalgebra of rooted trees

Connes and Kreimer [2] introduced a Lie algebra additively generated by rooted trees. They observed that the Lie bracket in this Lie algebra is obtained by antisymmetrization of a pre-Lie multiplication. We describe here the dual pre-Lie comultiplication.

By a *tree* we mean a finite tree. The set of edges of a tree T is denoted $\mathrm{edg}(T)$. A tree T with a distinguished vertex is *rooted*, the distinguished vertex being the *root* of T. A *homeomorphism* of rooted trees is a homeomorphism of trees mapping vertices onto vertices, edges onto edges, and the root into the root. Let $\mathcal{T}$ be the R-module freely generated by the set of homeomorphism classes of rooted trees.

Given an edge e of a rooted tree T, we obtain two other rooted trees as follows. Removing (the interior of) e from T, we obtain a graph $T - e$ having the same vertices as T. This graph consists of two disjoint trees T_e^1 and T_e^2 numerated so that the root of T lies on T_e^2. The root of T provides a root for T_e^2. As the root of T_e^1, we take the only vertex of T_e^1 adjacent to e in T.

Lemma 4.1.1. *The formula* $\rho(T) = \sum_{e\in \mathrm{edg}(T)} T_e^1 \otimes T_e^2$ *defines a pre-Lie comultiplication* $\rho : \mathcal{T} \to \mathcal{T}^{\otimes 2}$.

We shall prove a more general statement in the next subsection.

Antisymmetrizing ρ, we obtain a Lie cobracket $\nu : \mathcal{T} \to \mathcal{T}^{\otimes 2}$. Dualizing ρ and ν, we obtain a pre-Lie multiplication $\star$ and a Lie bracket in $\mathcal{T}^* = \mathrm{Hom}_R(\mathcal{T}, R)$. Note that the R-module $\mathcal{T}$ is based and therefore can be identified with the submodule of $\mathcal{T}^*$ consisting of those R-linear functionals $\mathcal{T} \to R$ which are non-zero only on a finite set of (homeomorphism classes of) rooted trees. More precisely, a rooted tree T is identified with the functional $\mathcal{T} \to R$ taking value 1 on T and value 0 on all other elements of the basis. It is easy to check that $\mathcal{T} \star \mathcal{T} \subset \mathcal{T} \subset \mathcal{T}^*$ so that $\mathcal{T}$ acquires the structure of a pre-Lie algebra. This structure and the associated Lie bracket were first defined in [2].

4.2. *Further pre-Lie coalgebras of trees*

The pre-Lie coalgebra $\mathcal{T}$ can be generalized using various additional structures on trees. We describe a general setting for such generalizations.

By a *subtree* of a tree T, we mean a tree $T' \subset T$ formed by a set of vertices and edges of T. If T is rooted then T' has a unique vertex v such that any path from the root of T to a point of T' passes through v. We take this v as the root of T'. In this way, all subtrees of a rooted tree become rooted.

We define a category *RTrees* whose objects are rooted trees and whose morphisms are embeddings. An *embedding* of rooted trees $j : T' \to T$ is a homeomorphism of T' onto a subtree of T. (Such j sends vertices, edges, and the root of T' onto vertices, edges, and the root of the subtree $j(T') \subset T$.) For example, homeomorphisms of rooted trees are embeddings.

A *rooted tree-structure* is a contravariant functor from the category *RTrees* to the category of sets. Such a functor φ assigns to any rooted tree T a set $\varphi(T)$ and to any embedding $j : T' \to T$ a map $\varphi(j) : \varphi(T) \to \varphi(T')$. We must have $\varphi(\mathrm{id}_T) = \mathrm{id}_{\varphi(T)}$ and $\varphi(jj') = \varphi(j')\varphi(j)$ for any embeddings $j' : T'' \to T'$ and $j : T' \to T$. We give two examples of a rooted tree-structure.

(a) (A labeling): The set $\varphi(T)$ consists of all labelings of vertices and edges of T by elements of certain sets S_0 and S_1, respectively. The map $\varphi(j) : \varphi(T) \to \varphi(T')$ is the obvious restriction of labelings.

(b) (A planar structure): The set $\varphi(T)$ consists of all topological embeddings $i : T \to \mathbb{R}^2$ considered up to composition with an orientation preserving homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. The map $\varphi(j) : \varphi(T) \to \varphi(T')$ is defined by $\varphi(j)(i) = ij$.

Given a rooted tree-structure φ, we define a *rooted φ-tree* to be a pair (a rooted tree T, an element $s \in \varphi(T)$). Two such pairs (T, s), $(\tilde{T}, \tilde{s})$ are homeomorphic if there is a homeomorphism $j : T \to \tilde{T}$ such that $\varphi(j)(\tilde{s}) = s$. For a subtree $T' \subset T$, set $s|_{T'} = \varphi(j)(s) \in \varphi(T')$, where j is the embedding $T' \hookrightarrow T$.

For any rooted tree-structure φ, we define a *(generalized) Connes–Kreimer pre-Lie coalgebra* $(\mathcal{T}(\varphi), \rho)$. Here, $\mathcal{T}(\varphi)$ is the R-module freely generated by the set of homeomorphism classes of rooted φ-trees. (To emphasize the dependence of R we shall sometimes denote this module by $\mathcal{T}(\varphi; R)$.)

Lemma 4.2.1. *The following formula defines a pre-Lie comultiplication ρ in $\mathcal{T}(\varphi)$:*

$$\rho(T, s) = \sum_{e \in \mathrm{edg}(T)} (T_e^1, s|_{T_e^1}) \otimes (T_e^2, s|_{T_e^2}) \tag{4.2.1}$$

Proof. We must verify Eq. (2.2.2) for $A = \mathcal{T}(\varphi)$. Pick a generator (T, s) of $\mathcal{T}(\varphi)$. A direct application of definitions shows that both $(\rho \otimes \mathrm{id})\rho(T, s)$ and $(\mathrm{id} \otimes \rho)\rho(T, s)$ are sums of certain expressions numerated by pairs $e_1, e_2 \in \mathrm{edg}(T)$. Pick such a pair e_1, e_2. The complement of the (interiors of) these two edges in T consists of three disjoint subtrees $T_0, T_1, T_2 \subset T$, where the notation is chosen so that e_k connects a vertex of T_0 with a vertex of T_k for $k = 1, 2$. Set $a_k = (T_k, s|_{T_k}) \in \mathcal{T}(\varphi)$, where $k = 0, 1, 2$. We consider three cases depending on whether the root v of T lies in T_0, T_1, or T_2. If $v \in T_0$, then the contributions of the pair e_1, e_2 to $(\rho \otimes \mathrm{id})\rho(T, s)$ and $(\mathrm{id} \otimes \rho)\rho(T, s)$ are, respectively, 0 and $a_1 \otimes a_2 \otimes a_0 + a_2 \otimes a_1 \otimes a_0$. If $v \in T_1$, then the contributions of the pair e_1, e_2 to both $(\rho \otimes \mathrm{id})\rho(T, s)$ and $(\mathrm{id} \otimes \rho)\rho(T, s)$ are equal to $a_2 \otimes a_0 \otimes a_1$. The case $v \in T_2$ is similar. In all cases, the contribution of e_1, e_2 to $((\rho \otimes \mathrm{id})\rho - (\mathrm{id} \otimes \rho)\rho)(T, s)$ is invariant under the permutation of the first two tensor factors. This gives (2.2.2). $\square$

Antisymmetrizing ρ, we obtain a Lie cobracket ν in $\mathcal{T}(\varphi)$. Dualizing ρ and ν, we obtain a pre-Lie multiplication $\star$ and a Lie bracket in $\mathcal{T}(\varphi)^*$. Forgetting the tree-structure yields, a pre-Lie (and Lie) coalgebra homomorphism $\mathcal{T}(\varphi) \to \mathcal{T}$ and the adjoint pre-Lie (and Lie) algebra homomorphism $\mathcal{T}^* \to \mathcal{T}(\varphi)^*$.

We can identify $\mathcal{T}(\varphi)$ with the submodule of $\mathcal{T}(\varphi)^*$ consisting of functionals which are non-zero only on a finite set of (homeomorphism classes of) rooted φ-trees. If the set $\varphi(T)$ is finite for any T, then $\mathcal{T}(\varphi) \star \mathcal{T}(\varphi) \subset \mathcal{T}(\varphi) \subset \mathcal{T}(\varphi)^*$ so that $\mathcal{T}(\varphi)$ acquires the structure of a pre-Lie (and Lie) algebra. The forgetting homomorphism $\mathcal{T}^* \to \mathcal{T}(\varphi)^*$ induces then a pre-Lie (and Lie) algebra homomorphism from $\mathcal{T} \subset \mathcal{T}^*$ to $\mathcal{T}(\varphi) \subset \mathcal{T}(\varphi)^*$. It sends a rooted tree T to $\sum_{s \in \varphi(T)} (T, s)$.

Lemma 4.1.1 follows from Lemma 4.2.1 by taking as φ the tree-structure assigning a one-element set to every rooted tree.

5. From loops to trees

5.1. Homomorphism η

We construct a canonical pre-Lie coalgebra homomorphism $\eta : \mathcal{L} \to \mathcal{T}(\Phi)$ for an appropriate rooted tree-structure $\Phi = \Phi_\Sigma$. Here, Σ is an oriented surface, $\mathcal{L} = \mathcal{L}(\Sigma)$, and Φ is a combination of a labeling and a planar structure. For a rooted tree T, the set $\Phi(T)$ consists of the triples (a labeling of the edges of T by ± 1, a labeling of the vertices of T by isotopy classes of (non-pointed) loops on Σ, a planar structure on T). In other words, the module $\mathcal{T}(\Phi)$ is generated by planar rooted trees whose edges are labeled with a sign and whose vertices are labeled with loops on Σ. Note that forgetting some (or all) of these additional structures on rooted trees we obtain homomorphisms from $\mathcal{L}$ to other Connes–Kreimer pre-Lie coalgebras.

The definition of η goes as follows. Pick a loop $\alpha : S^1 \to \Sigma$. In Section 3, we associated with every crossing $p \in \#\alpha$ a segment $e_p \subset D^2$ with endpoints on $S^1 = \partial D^2$. We call a subset H of $\#\alpha$ a *cut* of α if $e_p \cap e_q = \emptyset$ for all distinct $p, q \in H$. For a cut $H \subset \#\alpha$, we write $H \prec \alpha$. With each such H, we associate a rooted Φ-tree T_H as follows. The segments $\{e_p\}_{p \in H}$ are mutually disjoint and split the unit disk D^2 into several convex regions called H-faces. The vertices of T_H are numerated by the H-faces. The edges of T_H are numerated by elements of H: the edge corresponding to $p \in H$ is denoted $[p]$ and connects the vertices of T_H corresponding to two H-faces adjacent to e_p. The graph T_H is dual to the splitting of D^2 into the H-faces. This graph can be embedded in D^2 as follows: each vertex is mapped into a point inside the corresponding H-face; each edge is mapped onto the straight segment connecting the images of its endpoints. It is clear from this description that T_H is a planar tree.

A vertex of T_H arising from an H-face V is labeled with the loop on Σ obtained as follows: moving along ∂V we apply α while we are on $\partial V \cap \partial D^2$. The key observation is that for all $p \in H$, the mapping $\alpha : S^1 \to \Sigma$ maps the endpoints of e_p to one and the same point, this ensures that our procedure gives a loop on Σ (well defined up to reparametrization).

It remains to provide T_H with a root and to assign signs to the edges of T_H. It is here that we need to assume that α is pointed with base point $*_\alpha \in \alpha(S^1)$. As the root of T_H, we take the vertex corresponding to the only H-face whose boundary contains the point $\alpha^{-1}(*_\alpha)$. (We call this H-face the *root face*.) We label each edge $[p]$ of T_H with the sign ε_p defined in Section 3.2 and set $\varepsilon_H = \prod_{p \in H} \varepsilon_p$. The resulting rooted Φ-tree is denoted T_H or $T_H(\alpha, *_\alpha)$. Set

$$\eta(\alpha) = \sum_{H \prec \alpha} \varepsilon_H \, T_H \in \mathcal{T}(\Phi) \tag{5.1.1}$$

where H runs over all cuts of α. This extends by R-linearity to a homomorphism $\eta : \mathcal{C} \to \mathcal{T}(\Phi)$.

Theorem 5.1.1. *η is a pre-Lie coalgebra homomorphism.*

Proof. Pick a pointed loop $(\alpha : S^1 \to \Sigma, *_\alpha \in \alpha(S^1))$. Then

$$\begin{aligned}(\eta \otimes \eta)(\rho(\alpha)) &= (\eta \otimes \eta)\left(\sum_{p \in \#\alpha} \varepsilon_p \, (\alpha_{p_1,p_2}, p) \otimes (\alpha_{p_2,p_1}, *_\alpha)\right)\\ &= \sum_{p \in \#\alpha} \sum_{H_1 \prec \alpha_{p_1,p_2}} \sum_{H_2 \prec \alpha_{p_2,p_1}} \varepsilon_p \, \varepsilon_{H_1} \varepsilon_{H_2} T_{H_1}(\alpha_{p_1,p_2}, p) \otimes T_{H_2}(\alpha_{p_2,p_1}, *_\alpha)\end{aligned}$$

Also

$$\rho\eta(\alpha) = \rho\left(\sum_{H\prec\alpha}\varepsilon_H T_H\right) = \sum_{H\prec\alpha}\varepsilon_H \sum_{e\in\mathrm{edg}(T_H)} (T_H)^1_e \otimes (T_H)^2_e$$

$$= \sum_{H\prec\alpha}\varepsilon_H \sum_{p\in H}(T_H)^1_{[p]} \otimes (T_H)^2_{[p]} = \sum_{p\in\#\alpha}\sum_{p\in H\prec\alpha}\varepsilon_H (T_H)^1_{[p]} \otimes (T_H)^2_{[p]}$$

Therefore, it suffices to prove that for every $p \in \#\alpha$,

$$\sum_{H_1\prec\alpha_{p_1,p_2}}\ \sum_{H_2\prec\alpha_{p_2,p_1}} \varepsilon_p\varepsilon_{H_1}\varepsilon_{H_2} T_{H_1}(\alpha_{p_1,p_2}, p) \otimes T_{H_2}(\alpha_{p_2,p_1}, *_\alpha)$$

$$= \sum_{p\in H\prec\alpha}\varepsilon_H (T_H)^1_{[p]} \otimes (T_H)^2_{[p]}$$

This equality follows from the existence of the bijective correspondence $(H_1, H_2) \mapsto \{p\} \cup H_1 \cup H_2$ between pairs of cuts $H_1 \prec \alpha_{p_1,p_2}$, $H_2 \prec \alpha_{p_2,p_1}$ and cuts $H \prec \alpha$ containing p. Under this correspondence, $\varepsilon_p\, \varepsilon_{H_1}\varepsilon_{H_2} = \varepsilon_H$, $T_{H_1}(\alpha_{p_1,p_2}, p) = (T_H)^1_{[p]}$, and $T_{H_2}(\alpha_{p_2,p_1}, *_\alpha) = (T_H)^2_{[p]}$. $\square$

5.2. Remarks

1. Each loop on Σ has a natural degree defined as the number of its self-intersections. This can be used to define an R-linear homomorphism $r : \mathcal{T}(\Phi; R) \to \mathcal{T}(\Phi; R[u])$, where $R[u]$ is the ring of 1-variable polynomials with coefficients in R. The homomorphism r sends a rooted Φ-tree T to $u^{|T|}T$, where $|T|$ is the total degree of T defined as the sum over the vertices of T of the degrees of the corresponding loops. It is clear that r is a pre-Lie coalgebra homomorphism and so is $r\eta : \mathcal{L} \to \mathcal{T}(\Phi; R[u])$. The latter homomorphism allows us to separate the terms of different total degrees in the expression for $\eta(\alpha)$. Quotienting $r\eta$ by u, we obtain a pre-Lie coalgebra homomorphism $r\eta(\mathrm{mod}\, u) : \mathcal{L} \to \mathcal{T}(\Phi; R)$ which is given by the same formula as η but with H running over all cuts of α such that $|T_H| = 0$. The equality $|T_H| = 0$ means that the loops labeling the vertices of T_H have no self-crossings. This can be rephrased by saying that H is a *maximal* cut of $\#\alpha$ not contained in a bigger cut.
2. In the definition of η, the tree-structure Φ can be lifted to a stronger tree-structure $\tilde{\Phi}$. Observe that the edges of a planar tree incident to a vertex v are cyclically ordered. A pair of consecutive edges is called a *corner* at v. The tree-structure $\tilde{\Phi}$ is formed by Φ and a choice of the corner at the root of the tree. For a subtree $T' \subset T$, the restriction mapping $\tilde{\Phi}(T) \to \tilde{\Phi}(T')$ is defined as follows. If T' contains the root v of T, then the distinguished corner of T' at v is the one that contains the distinguished corner of T at v. If T' does not contain the root of T then we distinguish the corner at the root of T' containing the only edge e of T such that $T' = T^2_e$. The trees T_H above all have a distinguished corner at the root, namely the corner containing the point $\alpha^{-1}(*_\alpha)$. This lifts η to a pre-Lie coalgebra homomorphism $\mathcal{L} \to \mathcal{T}(\tilde{\Phi})$.

3. The comultiplication ρ can be included in a family of pre-Lie comultiplications $\rho^{a,b}$ in $\mathcal{L}$ parametrized by pairs $a, b \in R$. To define $\rho^{a,b}$, we simply replace ε_p in the definition of ρ by $a + b\varepsilon_p$. Replacing ε_H by $\prod_{p\in H}(a + b\varepsilon_p)$ in the definition of η we obtain a pre-Lie coalgebra homomorphism from $(\mathcal{L}, \rho^{a,b})$ to $\mathcal{T}(\Phi)$. For $b = 0$, the pre-Lie coalgebra $(\mathcal{L}, \rho^{a,b})$ is independent of the choice of orientation in Σ and can be defined for non-orientable surfaces.
4. Cuts on loops were introduced in [8], Section 15, where they are used to relate loops on Σ to knots in $\Sigma \times \mathbb{R}$.
5. Non-generic loops on Σ also lead to interesting and quite involved algebraic structures. The author plans to study them elsewhere.
6. The work of Chas and Sullivan [1] suggests that the constructions of this paper generalize to loops in manifolds of arbitrary dimension.

6. Weaker Lie coalgebras

In this section, we address the following question: can one define algebraic coproducts as above under weaker assumptions on loops and trees? Specifically, we are interested in non-pointed loops and oriented but non-rooted trees. The pre-Lie comultiplications defined above do not survive in this setting. However, as we show here, the associated Lie cobrackets do survive.

6.1. Loops re-examined

Forgetting the base points in the definition of isotopy of loops on an oriented surface Σ, we obtain isotopy for (non-pointed) loops. Denote by $\mathcal{L}_0$ the R-module freely generated by the set of isotopy classes of (non-pointed) loops on Σ. Forgetting the base point, yields a projection $\mathrm{pr} : \mathcal{L} \to \mathcal{L}_0$.

Lemma 6.1.1. *The Lie cobracket ν in $\mathcal{L}$ induces a Lie cobracket ν_0 in $\mathcal{L}_0$.*

Proof. We need to prove that when we forget the base points of loops on the right-hand side of formula (3.2.2), the resulting expression does not depend on the choice of $*_\alpha$. The reason for this comes from the fact that for each $p \in \#\alpha$, the two points of the set $\alpha^{-1}(p)$ have a natural order p^1, p^2 independent of $*_\alpha$. This order is defined by the condition that the pair (the positive tangent direction of α at p^1, the positive tangent direction of α at p^2) is positive with respect to the orientation of Σ. If α is pointed then $p^1 = p_1$, $p^2 = p_2$ in the case $\varepsilon_p = 1$ and $p^1 = p_2$, $p^2 = p_1$ in the case $\varepsilon_p = -1$. In all cases,

$$\varepsilon_p(\alpha_{p_1,p_2} \otimes \alpha_{p_2,p_1} - \alpha_{p_2,p_1} \otimes \alpha_{p_1,p_2}) = \alpha_{p^1,p^2} \otimes \alpha_{p^2,p^1} - \alpha_{p^2,p^1} \otimes \alpha_{p^1,p^2}$$

We can thus write down an explicit formula for ν_0:

$$\nu_0(\alpha) = \sum_{p\in\#\alpha} \alpha_{p^1,p^2} \otimes \alpha_{p^2,p^1} - \alpha_{p^2,p^1} \otimes \alpha_{p^1,p^2} \qquad \square \tag{6.1.1}$$

6.2. Lie coalgebra of oriented trees

An *oriented tree* is a tree with oriented edges. Any subtree of an oriented tree is oriented in the obvious way. We define a category *OTrees* whose objects are oriented trees and whose morphisms are orientation preserving embeddings (mapping vertices to vertices and edges to edges). An *oriented tree-structure* is a contravariant functor ψ from the category *OTrees* to the category of sets. Given an oriented tree-structure ψ, an *oriented ψ-tree* is a pair (an oriented tree T, an element $t \in \psi(T)$). Two such pairs (T, t), $(\tilde{T}, \tilde{t})$ are *homeomorphic* if there is a homeomorphism $j : T \to \tilde{T}$ such that $\psi(j)(\tilde{t}) = t$.

For any oriented tree-structure ψ, we define an R-module $\mathcal{T}_0(\psi)$ freely generated by the set of homeomorphism classes of oriented ψ-trees. Removing an edge e from an oriented tree T, we obtain two disjoint subtrees $T_e^1, T_e^2 \subset T$ numerated so that e is directed from a vertex of T_e^2 to a vertex of T_e^1.

Lemma 6.2.1. *For any oriented tree-structure ψ, the following formula defines a Lie cobracket in $\mathcal{T}_0(\psi)$:*

$$\nu_0(T, t) = \sum_{e \in \mathrm{edg}(T)} (T_e^1, t|_{T_e^1}) \otimes (T_e^2, t|_{T_e^2}) - (T_e^2, t|_{T_e^2}) \otimes (T_e^1, t|_{T_e^1})$$

This is proven along the same lines as Lemma 4.2.1; the difference is that instead of various positions of the root one has to consider four possible orientations on e_1, e_2. (The identity used in the proof of Lemma 2.2.1 and a similar identity with $P^{1,2}$ replaced by $P^{2,3}$ may help to shorten the computations). Warning: the homomorphism $\mathcal{T}_0(\psi) \to \mathcal{T}_0(\psi)^{\otimes 2}$ defined by $(T, t) \mapsto \sum_e (T_e^1, t|_{T_e^1}) \otimes (T_e^2, t|_{T_e^2})$ is *not* a pre-Lie cobracket.

If the set $\psi(T)$ is finite for all T, then the Lie cobracket ν_0 induces a Lie bracket in $\mathcal{T}_0(\psi)$ using the standard embedding $\mathcal{T}_0(\psi) \hookrightarrow \mathcal{T}_0(\psi)^*$.

Every rooted tree admits a canonical orientation uniquely defined by the condition that all edges adjacent to the root are outgoing and all other vertices are adjacent to exactly one incoming edge. This defines a covariant functor $h : RTrees \to OTrees$.

We shall be particularly interested in the oriented tree-structure $\Psi = \Psi_\Sigma$ assigning to an oriented tree T the set of pairs (a labeling of the vertices of T by isotopy classes of (non-pointed) loops on Σ, a planar structure on T). Let $\Phi = \Phi_\Sigma$ be the rooted tree-structure defined in Section 5.1. For a rooted tree T and $s \in \Phi(T)$, let $\mathrm{sign}_s(T)$ be the product of the signs labeling the edges of T. Let $h_s(T)$ be the oriented tree obtained from $h(T)$ by inverting orientation on all edges labeled with -1. The Φ-structure s induces a Ψ-structure s' on $h_s(T)$ by keeping the labels of the vertices and the embedding into $\mathbb{R}^2$.

We define an R-linear homomorphism $\mathrm{pr}_{\mathcal{T}} : \mathcal{T}(\Phi) \to \mathcal{T}_0(\Psi)$ by $\mathrm{pr}_{\mathcal{T}}(T, s) = \mathrm{sign}_s(T)(h_s(T), s')$.

Lemma 6.2.2. *The homomorphism $\mathrm{pr}_{\mathcal{T}} : \mathcal{T}(\Phi) \to \mathcal{T}_0(\Psi)$ is a Lie coalgebra homomorphism.*

The proof is an exercise on the definitions.

6.3. Homomorphism η_0

We define a Lie coalgebra homomorphism $\eta_0 : \mathcal{L}_0 \to \mathcal{T}_0(\Psi)$ (a version of η for non-pointed loops). For a loop $\alpha : S^1 \to \Sigma$, set

$$\eta_0(\alpha) = \sum_{H \prec \alpha} T_H \in \mathcal{T}_0(\Psi)$$

where T_H is the planar (non-rooted) tree determined by H. The labels of the vertices of T_H are as in Section 5.1. The edges of T_H are oriented as follows. For a crossing $p \in \#\alpha$, we orient the segment $e_p \subset D^2$ from p^1 to p^2 (in the notation introduced in the proof of Lemma 6.1.1) and orient the edge $[p] \subset T_H \subset D^2$ so that the pair $([p], e_p)$ determines the counterclockwise orientation of D^2. (By the definition of $[p]$, it intersects e_p transversally in one point). The next lemma follows directly from the definitions.

Lemma 6.3.1. *The following diagram is commutative*:

$$\begin{array}{ccc} \mathcal{C} & \stackrel{\mathrm{pr}}{\to} & \mathcal{C}_0 \\ \eta \downarrow & & \downarrow \eta_0 \\ \mathcal{T}(\Phi) & \stackrel{\mathrm{pr}}{\to} & \mathcal{T}_0(\Psi). \end{array}$$

Theorem 6.3.2. η_0 *is a Lie coalgebra homomorphism*.

Proof. By the results above η and $\mathrm{pr}_{\mathcal{T}}$ are Lie coalgebra homomorphisms. Hence, so is $\eta_0 \circ \mathrm{pr} = \mathrm{pr}_{\mathcal{T}} \circ \eta : \mathcal{L} \to \mathcal{T}_0(\Psi)$. Since $\mathrm{pr} : \mathcal{L} \to \mathcal{L}_0$ is a surjection, η_0 is a Lie coalgebra homomorphism. □

Remark 5.2.1 applies in this setting with obvious changes.

6.4. Related pre-Lie and Lie coalgebras

The constructions above can be adapted to so-called virtual strings, see [10]. An *open* (resp. *closed*) *virtual string of rank n* is a subset of $]0, 1[$ (resp. of S^1) consisting of $2n$ distinct points partitioned into n ordered pairs. These pairs are called *arrows*. The set of arrows of a virtual string a is denoted $\mathrm{arr}(a)$. Two open (resp. closed) virtual strings a, b are *homeomorphic* if there is an orientation preserving self-homeomorphism of $[0, 1]$ (resp. of S^1) transforming a into b.

Pick an arrow e of an open virtual string a with endpoints $p_1, p_2 \in]0, 1[$ numerated so that $p_1 < p_2$. Set $\varepsilon_e = 1$ if e is directed from p_1 to p_2 and $\varepsilon_e = -1$ otherwise. Denote by a_e^1 (resp. a_e^2) the virtual string obtained from a by removing e and all other arrows with at least one endpoint on $]0, 1[-[p_1, p_2]$ (resp. on $[p_1, p_2]$). The formula

$$\rho(a) = \sum_{e \in \mathrm{arr}(a)} \varepsilon_e a_e^1 \otimes a_e^2$$

defines a pre-Lie comultiplication in the R-module $\mathcal{S}$ freely generated by the set of homeomorphism classes of open virtual strings. This comultiplication is connected with obvious multiplication in $\mathcal{S}$ given by concatenation of open strings. Namely, $\rho(ab) = \rho(a)(1 \otimes b) + (1 \otimes a)\rho(b)$ for $a, b \in \mathcal{S}$.

Closed virtual strings can be obtained from the open ones by gluing 0 and 1. This gives a projection from $\mathcal{S}$ to the similar module $\mathcal{S}_0$ generated by homeomorphism classes of closed virtual strings. The pre-Lie comultiplication does not survive this operation but the associated Lie cobracket survives. The homomorphisms η and η_0 have their analogues: a pre-Lie coalgebra homomorphism $\mathcal{S} \to \mathcal{T}(\Phi')$ and a Lie coalgebra homomorphism $\mathcal{S}_0 \to \mathcal{T}_0(\Psi')$, where Φ' is a rooted tree-structure combining a labeling of edges by ± 1 with a planar structure and Ψ' is a planar structure.

Finally, observe that there is a projection from the coalgebras of loops on an oriented surface Σ into the coalgebras of virtual strings. The key observation is that every pointed loop α on Σ determines an open virtual string $a(\alpha)$ formed by the ordered pairs (p^1, p^2) with $p \in \#\alpha$. Here, we identify $S^1 - \alpha^{-1}(*_\alpha)$ with $]0, 1[$ via an orientation preserving homeomorphism. The formula $\alpha \mapsto a(\alpha)$ defines a pre-Lie coalgebra homomorphism $\mathcal{L}(\Sigma) \to \mathcal{S}$. It is in general neither surjective nor injective. In particular, pointed loops on Σ related by the action of the mapping class group have the same images in $\mathcal{S}$. The homomorphism $\mathcal{L}(\Sigma) \to \mathcal{S}$ induces a Lie coalgebra homomorphism $\mathcal{L}_0(\Sigma) \to \mathcal{S}_0$. We also have the obvious forgetting homomorphisms $\mathcal{T}(\Phi) \to \mathcal{T}(\Phi')$ and $\mathcal{T}_0(\Psi) \to \mathcal{T}_0(\Psi')$ making all the natural diagrams arising here commutative.

7. Lie bialgebra of loops

We relate the Lie coalgebra $\mathcal{L}_0 = \mathcal{L}_0(\Sigma)$ to the Lie bialgebra of loops on Σ introduced in [7,8].

7.1. *Lie coalgebra $Z = Z(\Sigma)$*

Loops α, β on Σ are *freely homotopic* if there is a mapping $f : S^1 \times [0, 1] \to \Sigma$ such that $\alpha(a) = f(a, 0)$ and $\beta(a) = f(a, 1)$ for all $a \in S^1$. Free homotopy is an equivalence relation on the set of loops. The corresponding set of equivalence classes is denoted $\hat{\pi} = \hat{\pi}(\Sigma)$. This set has a distinguished element α_0 represented by an embedding $S^1 \hookrightarrow \Sigma$ onto the boundary of a small disk in Σ. For connected Σ, the set $\hat{\pi}$ can be identified with the set of conjugacy classes in the fundamental group π of Σ.

Let Z be the R-module freely generated by the set $\hat{\pi}$. Since isotopic loops are homotopic, assigning to an isotopy class of loops the underlying homotopy class we obtain an R-linear homomorphism $P : \mathcal{L}_0 \to Z$. The Lie cobracket ν_0 in $\mathcal{L}_0$ cannot directly induce a Lie cobracket in Z because of the following obstruction. Consider a loop $\alpha : S^1 \to \Sigma$ and insert a small φ-like cirl on the right of α. This gives a new loop, α', homotopic to α. It is clear from formula (6.1.1) that

$$(P \otimes P)\nu_0(\alpha') = (P \otimes P)\nu(\alpha) + \alpha_0 \otimes P(\alpha) - P(\alpha) \otimes \alpha_0 \neq (P \otimes P)\nu_0(\alpha)$$

This obstruction can be circumvent as follows. Let $g : Z \to Z$ be the R-linear endomorphism defined by $g(a) = a$ for all $a \in \hat{\pi} - \{\alpha_0\}$ and $g(\alpha_0) = 0$.

Lemma 7.1.1. ([7,8]). *The following formula defines a Lie cobracket* $\nu_Z : Z \to Z^{\otimes 2}$:

$$\nu_Z(\alpha) = (g \otimes g)\left(\sum_{p \in \#\alpha} \alpha_{p^1,p^2} \otimes \alpha_{p^2,p^1} - \alpha_{p^2,p^1} \otimes \alpha_{p^1,p^2}\right) \tag{7.1}$$

Formula (6.1.1) implies that $gP : \mathcal{L}_0 \to Z$ is a Lie coalgebra homomorphism. The map $\mathcal{L}_0 \to \mathcal{T}_0(\Psi)$ does not survive the factorization of loops by homotopy: the linear combination of trees associated with a loop may change drastically under homotopy. However, there appears another fundamental structure described next.

7.2. *Goldman's Lie bracket in Z*

Goldman [5] defined a Lie bracket [,] in Z as follows. (A related Lie algebra is implicit in the earlier paper of Wolpert [12].) Let α, β be two loops on Σ. Applying a small isotopy to α we can assume that α meets β transversally at a finite number of points distinct from self-intersections of α, β. Denote the (finite) set $\alpha(S^1) \cap \beta(S^1)$ by $\alpha\#\beta$. Each point $p \in \alpha\#\beta$ is a double transversal intersection of α and β. Let $(\alpha \cdot \beta)_p = \pm 1$ denote the intersection index of α and β at p. Smoothing the set $\alpha(S^1) \cup \beta(S^1)$ at p we obtain a loop on Σ denoted $(\alpha\beta)_p$. This smoothing replaces the X-like crossing at p by two disjoint arcs $\uparrow\uparrow$ so that arriving to a neighborhood of p along α (resp. β) one leaves along β (resp. α). Set

$$[\alpha, \beta] = \sum_{p \in \alpha\#\beta} (\alpha \cdot \beta)_p (\alpha\beta)_p$$

Extending by bilinearity we obtain a bracket [,] in Z.

Theorem 7.2.1. [5] [,] *is a well defined Lie bracket in Z.*

To explain the connection between the Lie cobracket ν_Z and Goldman's Lie bracket, we recall the notion of a Lie bialgebra due to V. Drinfeld. A *Lie bialgebra* over R is an R-module A endowed with a Lie bracket [,] and a Lie cobracket $\nu : A \to A^{\otimes 2}$ such that $\nu([x, y]) = x\nu(y) - y\nu(x)$ for any $x, y \in A$. Here, A acts on $A \otimes A$ by $x(y \otimes z) = [x, y] \otimes z + y \otimes [x, z]$.

Theorem 7.2.2. ([7,8]). *The triple* $(Z, [\,,\,], \nu_Z)$ *is a Lie bialgebra.*

This bialgebra has a topological quantization (in fact, a biquantization) in terms of a Hopf algebra of skein classes of oriented links in $\Sigma \times \mathcal{R}$. It is curious to note that this algebra acts on the spaces of conformal blocks associated with Σ by appropriate two-dimensional modular functors.

8. Hopf algebras of trees and loops

8.1. Symmetric algebras

Given an R-module A, one has its symmetric algebra

$$S(A) = \oplus_{n\geq 0} S^n(A)$$

where $S^0(A) = R$, $S^1(A) = A$, and $S^n(A)$ is the nth symmetric tensor power of A for $n \geq 2$. The algebra $S(A)$ is commutative and associative and has a unit $1 \in R = S^0(A)$. The projection $S(A) \to S^0(A) = R$ along $\oplus_{n\geq 1} S^n(A)$ is called the *augmentation*.

If A is a free module with basis $\{x_i\}_i$, then $S(A)$ can be identified with the polynomial algebra $R[\{x_i\}_i]$.

8.2. Connes–Kreimer Hopf algebras

Consider the symmetric algebra $S(\mathcal{T})$ whose elements are polynomials on rooted trees with coefficients in R. (The unit $1 \in S^0(\mathcal{T})$ can be thought of as an empty tree.) Connes and Kreimer [2] defined a non-cocommutative comultiplication in $S(\mathcal{T})$ which makes it into a bialgebra. We recall their definition extending it (in a straightforward way) to the setting of rooted trees with structure. Fix a rooted tree-structure φ. A *simple cut* of a rooted tree T is a set $c \subset \mathrm{edg}(T)$ such that any embedded path leading from the root of T to a vertex of T meets at most one element of c. Removing from T all (open) edges belonging to a simple cut c we obtain a set of disjoint subtrees of T. One of them denoted T_0 contains the root of T. The other subtrees $\{T_e\}_{e\in c}$ are numerated by the elements of c so that each $e \in c$ connects a vertex of T_0 to a vertex of T_e. Recall that all subtrees of T are rooted in a canonical way. For $s \in \varphi(T)$, set

$$l_c(T, s) = \prod_{e\in c} (T_e, s|_{T_e}) \in S(\mathcal{T}(\varphi)),\ r_c(T, s) = (T_0, s|_{T_0}) \in S^0(\mathcal{T}(\varphi))$$

Set

$$\nabla(T, s) = (T, s) \otimes 1 + \sum_c l_c(T, s) \otimes r_c(T, s) \tag{8.2.1}$$

where c runs over all simple cuts of T. Note that the term $l_c \otimes r_c$ corresponding to $c = \emptyset$ is equal to $1 \otimes (T, s)$. Formula 8.2.1 defines ∇ on the generators of the algebra $S = S(\mathcal{T}(\varphi))$; it extends uniquely to an algebra homomorphism $\nabla : S \to S \otimes S$. It follows from the definitions that the augmentation $S \to R$ is a counit of ∇. Connes and Kremer proved that ∇ is coassociative. They also explain that the resulting bialgebra $S(\mathcal{T}(\varphi))$ has an antipode and is thus a Hopf algebra.

8.3. Hopf algebra of pointed loops

Consider the symmetric algebra $S = S(\mathcal{L})$ where $\mathcal{L} = \mathcal{L}(\Sigma)$ is the R-module freely generated by isotopy classes of pointed loops on an oriented surface Σ. Elements of S are polynomials on isotopy classes of pointed loops on Σ with coefficients in R. We define a comultiplication Δ in S as follows. Pick a pointed loop α on Σ. Recall the segments $\{e_p\}_{p\in\#\alpha}$ in the unit disk D and the H-faces of D determined by a cut $H \prec \alpha$, cf. Sections 3.1 and 5.1. Let $v(H)$ denote the root H-face, i.e., the only H-face containing $\alpha^{-1}(*_\alpha) \in \partial D$. For $p \in H$ denote by $v(H, p)$ the unique H-face adjacent to e_p and such that $v(H, p)$ and $v(H)$ lie on different sides of the line containing e_p. The formula $p \mapsto v(H, p)$ establishes a bijection between H and the set of H-faces distinct from $v(H)$. For any H-face v denote by α_v the associated pointed loop on Σ. Set

$$l_H(\alpha) = \prod_{p\in H} \alpha_{v(H,p)} \in S,\ r_H(\alpha) = \alpha_{v(H)} \in \mathcal{L} \subset S$$

A cut H of α is *simple* if all segments $\{e_p\}_{p\in H}$ are adjacent to $v(H)$. To indicate that H is a simple cut of α we write $H \ll \alpha$. Set

$$\Delta(\alpha) = \alpha \otimes 1 + \sum_{H\ll\alpha} \varepsilon_H\, l_H(\alpha) \otimes r_H(\alpha) \in S \otimes S$$

where $\varepsilon_H = \prod_{p\in H} \varepsilon_p$. Note that the term $\varepsilon_H\, l_H(\alpha) \otimes r_H(\alpha)$ corresponding to $H = \emptyset$ is equal to $1 \otimes \alpha$.

This defines Δ on the generators of S; it extends uniquely to an algebra homomorphism $\Delta : S \to S \otimes S$.

Lemma 8.3.1. *Δ is coassociative.*

Proof. It suffices to prove that $(\mathrm{id} \otimes \Delta)\Delta(\alpha) = (\Delta \otimes \mathrm{id})\Delta(\alpha)$ for any pointed loop α on Σ. Set

$$A = (\mathrm{id} \otimes \Delta)\Delta(\alpha) - \Delta(\alpha) \otimes 1,\ B = (\Delta \otimes \mathrm{id})\Delta(\alpha) - \Delta(\alpha) \otimes 1.$$

We shall prove that $A = B$. To this end, we define another expression C and prove that $A = C = B$.

For simple cuts $G \ll \alpha$, $G' \ll \alpha$, we write $G' \leq G$ if $v(G\alpha) \subset v(G)$. Then, $G \cup G' \subset \#\alpha$ is a cut of α. Its faces are the G-faces $\{v(G \cup G', p) = v(G, p)\}_{p\in G}$ and the faces obtained by splitting $v(G)$ along the segments $\{e_q\}_{q\in G'-G}$, specifically, $\{v(G \cup G', q)\}_{q\in G'-G}$ and $v(G')$. (Note that $G' - G = G' - (G \cap G')$.) Set

$$C = \sum_{G,G'\ll\alpha\, s.t.\, G'\leq G} \varepsilon_{G\cup G'} \prod_{p\in G} \alpha_{v(G,p)} \otimes \prod_{q\in G'-G} \alpha_{v(G\cup G',q)} \otimes \alpha_{v(G')} \in S^{\otimes 3}$$

We claim that $A = C$. If follows from the definition of Δ that

$$A = \sum_{H\ll\alpha} \varepsilon_H \sum_{H'\ll r_H(\alpha)} \varepsilon_{H'} l_H(\alpha) \otimes l_{H'}(r_H(\alpha)) \otimes r_{H'}(r_H(\alpha))$$

The formula $(G, G') \mapsto (H = G, H' = G' - G)$ defines a bijective correspondence between pairs $(G \ll \alpha, G' \ll \alpha)$ such that $G' \leq G$ and pairs $(H \ll \alpha, H' \ll r_H(\alpha))$. The corresponding terms of A and C are equal. The equality of signs follows from the formula $\varepsilon_{G\cup G'} = \varepsilon_G\, \varepsilon_{G'-G} = \varepsilon_H \varepsilon_{H'}$. Therefore, $A = C$.

We claim that $B = C$. If follows from the definition of Δ that

$$B = \sum_{H \ll \alpha} \varepsilon_H \prod_{p\in H} \left(\alpha_{v(H,p)} \otimes 1 + \sum_{H_p \ll \alpha_{v(H,p)}} \varepsilon_{H_p} l_{H_p}(\alpha_{v(H,p)}) \otimes r_{H_p}(\alpha_{v(H,p)}) \right) \otimes r_H(\alpha)$$

$$= \sum_{H \ll \alpha} \varepsilon_H \sum_{I \subset H} \sum_{\{H_p \ll \alpha_{v(H,p)}\}_{p\in H-I}} \varepsilon_{\cup_p H_p} \left(\prod_{q\in I} \alpha_{v(H,q)} \prod_{p\in H-I} l_{H_p}(\alpha_{v(H,p)}) \right)$$

$$\otimes \prod_{p\in H-I} r_{H_p}(\alpha_{v(H,p)}) \otimes r_H(\alpha)$$

With each tuple $(H \ll \alpha, I \subset H, \{H_p \ll \alpha_{v(H,p)}\}_{q\in H-I})$, we associate the pair $(G = I \cup \cup_{p\in H-I} H_p, G' = H)$. This defines a bijective correspondence between such tuples and the pairs $(G \ll \alpha, G' \ll \alpha)$ such that $G' \leq G$. The corresponding terms of B and C are equal. The equality of signs follows from the formula $G \cup G' = H \cup \cup_{p\in H-I} H_p$ and the fact that the sets $\{H_p\}_{p\in H-I}$ and H are pairwise disjoint. Therefore, $B = C$. $\square$

It is clear that the augmentation $\varepsilon : S \to R$ is a counit of Δ. The bialgebra (S, Δ) has an antipode s. This is an algebra endomorphism of S determined on a generator α by induction on $|\alpha| = \mathrm{card}(\#\alpha)$: if $|\alpha| = 0$, then $s(\alpha) = -\alpha$, if $|\alpha| \geq 1$, then

$$s(\alpha) = -\alpha - \sum_{H \ll \alpha, H \neq \emptyset} \varepsilon_H\, l_H(\alpha)\, s(r_H(\alpha)) \in S$$

where we use that $|r_H(\alpha)| < |\alpha|$. These formulas guarantee that $m(\mathrm{id}_S \otimes s)\Delta(\alpha) = \varepsilon(\alpha)$, where m is multiplication in S. In other words, s is a left inverse of id_S with respect to the convolution product $\star$ in $\mathrm{Hom}_R(S, S)$ defined by $f \star g = m(f \otimes g)\Delta$ for $f, g \in \mathrm{Hom}_R(S, S)$. Similar inductive formulas show that id_S has a right inverse $s' \in \mathrm{Hom}_R(S, S)$ and then $s = s \star (\mathrm{id}_S \star s') = (s \star \mathrm{id}_S) \star s' = s'$. Therefore, s is an antipode for S.

8.4. Homomorphism η

The R-linear homomorphism $\eta : \mathcal{L} \to \mathcal{T}(\Phi)$ defined in Section 5.1 extends by multiplicativity to an algebra homomorphism $S(\mathcal{L}) \to S(\mathcal{T}(\Phi))$ also denoted η.

Theorem 8.4.1. *η is a Hopf algebra homomorphism.*

Proof. We need to show that $\nabla(\eta(\alpha)) = (\eta \otimes \eta)(\Delta(\alpha))$ for any pointed loop $\alpha : S^1 \to \Sigma$. Set $a = \nabla(\eta(\alpha)) - \eta(\alpha) \otimes 1$ and $b = (\eta \otimes \eta)(\Delta(\alpha) - \alpha \otimes 1)$. It is enough to check that $a = b$.

Consider a cut $H \subset \#\alpha$ of α and a subset $G \subset H$ such that G is a simple cut of α. The cut G determines a simple cut $c(G, H)$ of the tree $T_H = T_H(\alpha)$ consisting of the edges

$\{\langle p\rangle\}_{p\in G}$. (This establishes a bijection between simple cuts of T_H and subsets of H which are simple cuts of α.) Set

$$\langle G, H\rangle = l_{c(G,H)}(T_H) \otimes r_{c(G,H)}(T_H) \in S(\mathcal{T}(\Phi)) \otimes \mathcal{T}(\Phi).$$

By abuse of notation we do not specify the Φ-structure on the trees on the right-hand side; it is induced by the one on T_H. It is easy to deduce from the definitions that

$$b = \sum_{G\ll\alpha} \varepsilon_G \sum_{G\subset H\prec\alpha} \varepsilon_{H-G}\, \langle G, H\rangle = \sum_{H\prec\alpha} \varepsilon_H \sum_{G\subset H\, s.t.\, G\ll\alpha} \langle G, H\rangle = a.$$

Thus, η is a bialgebra homomorphism. Finally, any bialgebra homomorphism of Hopf algebras is a Hopf algebra homomorphism, see [6], Lemma 4.0.4. $\square$

8.5. Non-commutative Hopf algebra of loops

Given an R-module A, one has its tensor algebra $T(A) = \oplus_{n\geq 0} A^{\otimes n}$, where $A^{\otimes 0} = R$, $A^{\otimes 1} = A$, and $A^{\otimes n}$ with $n \geq 2$ is the tensor product over R of n copies of A. The product in $T(A)$ is defined by

$$(a_1 \otimes \cdots \otimes a_n)(a_{n+1} \otimes \cdots \otimes a_{n+m}) = a_1 \otimes \cdots \otimes a_{n+m}$$

for $a_1, \ldots, a_{n+m} \in A$. In the sequel instead of $a_1 \otimes \cdots \otimes a_n$, we write $\prod_i a_i$. The algebra $T(A)$ is associative and has a unit $1 \in R = A^{\otimes 0}$. The identity map $A \to A$ extends to a surjective algebra homomorphism $T(A) \to S(A)$. If A is a free module with basis $\{x_i\}_i$, then $T(A)$ is the algebra of non-commutative polynomials in the variables $\{x_i\}_i$ with coefficients in R.

Consider the tensor algebra $T = T(\mathcal{L})$, where $\mathcal{L} = \mathcal{L}(\Sigma)$. Observe that any simple cut $H \subset \#\alpha$ of a pointed loop α is totally ordered in a canonical way. Namely starting at the base point $*_\alpha$ and moving along the loop we meet first a certain point of H twice, then another point of H twice, etc. The resulting order on H allows us to set $\tilde{l}_H(\alpha) = \prod_{p\in H} \alpha_{v(H,p)} \in \mathcal{L}^{\otimes n} \subset T$. The formula

$$\tilde{\Delta}(\alpha) = \alpha \otimes 1 + \sum_{H\ll\alpha} \varepsilon_H\, \tilde{l}_H(\alpha) \otimes r_H(\alpha)$$

defines a map $\mathcal{L} \to T \otimes T$. It extends to an algebra homomorphism $\tilde{\Delta} : T \to T \otimes T$. The same argument as in the proof of Lemma 8.3 shows that $\tilde{\Delta}$ is coassociative. In this argument in the expressions for C, B, one should use the orders in $G' - G$ and $H - I$ (needed in the second tensor factor) induced by the orders in G' and H respectively. In the expression for B, one should replace $\prod_{q\in I} \alpha_{v(H,q)} \prod_{p\in H-I} l_{H_p}(\alpha_{v(H,p)})$ with $\prod_{p\in H} a_p$, where $a_p = \alpha_{v(H,p)}$ for $p \in I$ and $a_p = l_{H_p}(\alpha_{v(H,p)})$ for $p \in H - I$.

The projection $T \to R$ along $\oplus_{n\geq 1} \mathcal{L}^{\otimes n}$ is a counit of T. The existence of an antipode in T is straightforward. It is clear that the natural projection $T \to S(\mathcal{L})$ is a Hopf algebra homomorphism.

A non-commutative analogue of η is a Hopf algebra homomorphism $\tilde{\eta}$ from T to Foissy's [3] non-commutative Hopf algebra $\mathcal{H}_{P,R}(\Phi)$ generated by rooted Φ-trees. Note that Foissy considers planar rooted trees with labeled vertices but nothing prevents from extending his definitions to the case where the edges are also labeled. (Alternatively, one may observe that the edges of a rooted tree are numerated by the vertices distinct from the root so that a labeling of the edges can be interpreted as a labeling of the vertices.) The value of $\tilde{\eta}$ on the generators of $T = T(\mathcal{L})$ is given by formula (5.1.1). We have a commutative diagram of Hopf algebra homomorphisms

$$\begin{array}{ccc} T(\mathcal{L}) & \stackrel{\tilde{\eta}}{\to} & \mathcal{H}_{P,R}(\Phi) \\ \downarrow & & \downarrow \\ S(\mathcal{L}) & \stackrel{\eta}{\to} & S(\mathcal{T}(\Phi)) \end{array}$$

where the vertical arrows are the natural projections.

8.6. Remark

As in Remark 5.2.3, for any $a, b \in R$, we can replace everywhere (and in particular in the definition of ε_H) the sign ε_p with $a + b\varepsilon_p$. This yields a two-parameter family of coassociative comultiplications $\Delta^{a,b}$ in $S(\mathcal{L})$ (resp. $\tilde{\Delta}^{a,b}$ in $T(\mathcal{L})$) and Hopf algebra homomorphisms from the resulting Hopf algebras to $S(\mathcal{T}(\Phi))$ (resp. to $\mathcal{H}_{P,R}(\Phi)$).

9. Further algebras

In analogy with tree-structures, we can introduce axiomatically certain "structures" on loops suitable for a generalization of the comultiplications ρ, Δ defined above. Instead of doing this here, we focus on two specific additional structures on loops and briefly discuss the associated algebras.

9.1. Algebras of Wilson loops

By a *region* of a (generic) loop $\alpha : S^1 \to \Sigma$ on an oriented surface Σ, we mean a connected component of $\Sigma - \alpha(S^1)$. A *Wilson loop* is a loop on Σ whose regions are endowed with numbers (say, real or complex). The number associated with a region is called its *area*. A Wilson loop is *pointed* if its underlying geometric loop is pointed. Isotopy of (pointed) Wilson loops is defined in the obvious way, the areas being preserved under ambient isotopies and reparametrizations.

For a Wilson loop α and a crossing $p \in \#\alpha$, both loops α_{p_1,p_2} and α_{p_2,p_1} appearing in Section 3.2 become Wilson loops as follows. The area of a region X of α_{p_1,p_2} (resp. of α_{p_2,p_1}) is set to be the sum of the areas of regions of α contained in X.

Let $\mathcal{W}$ be the R-module freely generated by the set of isotopy classes of pointed Wilson loops. The definition of the pre-Lie comultiplication in Section 3.2 applies to Wilson loops word for word and gives a pre-Lie comultiplication in $\mathcal{W}$. In analogy with Section 6, the associated Lie cobracket in $\mathcal{W}$ induces a Lie cobracket in $\mathcal{W}_0$, the R-module freely generated by the set of isotopy classes of non-pointed Wilson loops. Forgetting the areas we obtain a pre-Lie coalgebra homomorphism $\mathcal{W} \to \mathcal{L}$ and a Lie coalgebra homomorphism $\mathcal{W}_0 \to \mathcal{L}_0$. Similarly, the definition of Δ in Section 8 applies to Wilson loops and gives Hopf algebra structures in $S(\mathcal{W})$ and $T(\mathcal{W})$ and a commutative diagram

$$\begin{array}{ccc} T(\mathcal{W}) & \to & T(\mathcal{L}) \\ \downarrow & & \downarrow \\ S(\mathcal{W}) & \to & S(\mathcal{L}) \end{array}$$

of surjective Hopf algebra homomorphisms. The comultiplications in $\mathcal{W}$, $S(\mathcal{W})$, $T(\mathcal{W})$ can be included in a 2-parameter family of comultiplications $\rho^{a,b}$, $\Delta^{a,b}$, $\tilde{\Delta}^{a,b}$ as in Remarks 5.2.3 and 8.6.

9.2. *Algebras of knot diagrams*

A *knot diagram* on an oriented surface Σ is a (generic) loop $\alpha : S^1 \to \Sigma$ such that each crossing $p \in \#\alpha$ is endowed with a sign $\mu_p = \pm 1$. The equivalence with the more standard language of over/undercrossings is established as follows. Recall that two branches of α passing through $p \in \#\alpha$ have an order determined by the orientation of Σ (cf. the proof of Lemma 6.1.1). Then, the first branch goes over (resp. under) the second branch if $\mu_p = 1$ (resp. if $\mu_p = -1$). Note that by the definition of a loop, our knot diagrams are oriented.

A knot diagram is *pointed* if its underlying geometric loop is pointed. Isotopy of (pointed) knot diagrams is defined in the obvious way, the signs μ being preserved under ambient isotopies and reparametrizations.

Let $\mathcal{D}$ be the R-module freely generated by the set of isotopy classes of pointed knot diagrams. Pick four elements $a, b, c, d \in R$. For a pointed knot diagram α and a crossing $p \in \#\alpha$, both loops α_{p_1,p_2} and α_{p_2,p_1} appearing in Section 3.2 become knot diagrams: their self-crossings are also self-crossings of α and we attribute to them the same signs μ. Set

$$\rho^{a,b,c,d}(\alpha, *_\alpha) = \sum_{p\in\#\alpha} (a + b\varepsilon_p + c\mu_p + d\varepsilon_p\mu_p)\,(\alpha_{p_1,p_2}, p) \otimes (\alpha_{p_2,p_1}, *_\alpha).$$

This defines a pre-Lie comultiplication $\rho^{a,b,c,d} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$. Note that multiplying all signs μ_p by -1 we obtain an isomorphism $(\mathcal{D}, \rho^{a,b,c,d}) \approx (\mathcal{D}, \rho^{a,b,-c,-d})$. For $c = d = 0$, this defines an involution in $(\mathcal{D}, \rho^{a,b,0,0})$.

Similarly, the definition of Δ in Section 8 can be applied to pointed knot diagrams and gives Hopf algebra comultiplications $\Delta^{a,b,c,d}$ in $S(\mathcal{D})$ and $\tilde{\Delta}^{a,b,c,d}$ in $T(\mathcal{D})$. (It is understood that we replace everywhere ε_p with $a + b\varepsilon_p + c\mu_p + d\varepsilon_p\mu_p$.) The definitions of η, $\tilde{\eta}$ also apply and yield a pre-Lie coalgebra homomorphism $(\mathcal{D}, \rho^{a,b,c,d}) \to \mathcal{T}(\Phi)$ and a commutative diagram of Hopf algebra homomorphisms

$$\begin{array}{ccc} (T(\mathcal{D}), \tilde{\Delta}^{a,b,c,d}) & \to & \mathcal{H}_{P,R}(\Phi) \\ \downarrow & & \downarrow \\ (S(\mathcal{D}), \Delta^{a,b,c,d}) & \to & S(\mathcal{T}(\Phi)). \end{array}$$

For $a = c = 0$, the Lie cobracket in $\mathcal{D}$ associated with $\rho^{a,b,c,d}$ induces a Lie cobracket in $\mathcal{D}_0$, the R-module freely generated by the set of isotopy classes of non-pointed knot diagrams. If additionally $d = 0$, $b = 1$, then we have a forgetting Lie coalgebra homomorphism $\mathcal{D}_0 \to \mathcal{L}_0$.

9.3. Homomorphisms

The pre-Lie algebras $\mathcal{L}$, $\mathcal{W}$, $\mathcal{D}$ are related by three pre-Lie algebra homomorphisms:

$$(\mathcal{L}, \rho^{a,b}) \to (\mathcal{D}, \rho^{a,b,0,0}) \to (\mathcal{W}, \rho^{a,b}) \to (\mathcal{L}, \rho^{a,b}). \tag{9.3.1}$$

We describe them on the generators. The leftmost homomorphism is obtained by attributing $\mu = +1$ to all crossings of a pointed loop. The rightmost homomorphism is obtained by forgetting the areas. The middle homomorphism $\mathcal{D} \to \mathcal{W}$ comes from the theory of shadow knots [9]. It transforms a pointed knot diagram (α, μ) into a pointed Wilson loop as follows. A crossing $p \in \#\alpha$ is adjacent to four (possibly coinciding) regions $R_1, \ldots, R_4$ of α which we numerate so that R_1 lies between the outgoing branches of α at p and R_3 lies between the incoming branches of α at p while R_2, R_4 are the two remaining regions. Then, p contributes $(-1)^{k+1}\mu_p/2$ to the area of R_k for $k = 1, \ldots, 4$. The area of a region of α is defined to be the sum of the contributions of the crossings of α adjacent to this region. The makes α into a pointed Wilson loop.

It is clear that the composition of the three homomorphisms in (9.3.1) is the identity map. These homomorphisms induce Hopf algebra homomorphisms

$$(S(\mathcal{L}), \Delta^{a,b}) \to (S(\mathcal{D}), \Delta^{a,b,0,0}) \to (S(\mathcal{W}), \Delta^{a,b}) \to (S(\mathcal{L}), \Delta^{a,b})$$

and similar Hopf algebra homomorphisms with S, Δ replaced by T, $\tilde{\Delta}$.

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